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INVERSE SUMS OF MONOTONE OPERATORS.(U)
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(9) MRC Technical Summary Report #2177

(16) INVERSE SUMS OF MONOTONE OPERATORS.

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(14) MRC-TSR-2177

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610 Walnut Street
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(15) DAAG29-80-C-0041, NSF-MCS79-01066

(11) Feb[redacted]81

(Received January 12, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

221200 81 5 27 013

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

INVERSE SUMS OF MONOTONE OPERATORS

Stephen M. Robinson

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February 1981

ABSTRACT

We show that the inverse image of a point under the sum of two monotone operators in R^n has a special form if a simple condition is imposed on one of the operators. This result is then applied to characterize the form of the solution set of a monotone linear generalized equation. Some known results are recovered as a special case.

AMS (MOS) Subject Classifications: 47H05, 90C33.

Key Words: Monotone operators, generalized equations, linear complementarity problem

Work Unit Number 5 (Operations Research)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and by the National Science Foundation under Grant No. MCS-7901066, Mod. 1.

SIGNIFICANCE AND EXPLANATION

Problems of importance in economics, structural engineering, and other areas can sometimes be expressed as generalized equations involving a special type of multivalued function called a monotone operator. The solutions of the problems are then the solutions of the generalized equation. Frequently the operator appearing in the generalized equation is the sum of two simpler operators.

In this paper we show that for such a sum, if one of the operators satisfies a simple condition which often holds in applications, then the generalized equation can be split into two generalized equations, each involving one of the simpler monotone operators together with a special single-valued continuous function. We investigate some properties of this function and then show how to use it to find the form of the solution set of a class of linear generalized equations.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

INVERSE SUMS OF MONOTONE OPERATORS

Stephen M. Robinson

1. INTRODUCTION.

This paper develops a property of sums of monotone operators in R^n. Such sums occur frequently in applications to such areas as linear and nonlinear programming, complementarity problems, etc. We show here that if the operators involved obey a rather simple condition, then the inverse image of a point under their sum is the intersection of the inverse images, under the individual operators, of two points related in a simple way to the original point. In Section 2 we investigate the nature of this relationship; then in Section 3 we apply it to characterize the structure of the solution sets of certain linear generalized equations. Some known results about solutions of positive semidefinite linear complementarity problems then appear as special cases of this characterization.

All of the results in this paper are stated for monotone operators from R^n to itself, primarily because the applications in Section 3 are in R^n. Some of this work could probably be extended to infinite-dimensional spaces if it were worthwhile to do so.

For ease of reference, we recall here that a (possibly multivalued) operator T is said to be <u>monotone</u> if for any pairs (x_1,y_1) and (x_2,y_2) in its graph $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$, where $\langle \cdot,\cdot \rangle$ denotes the standard inner product. We say T is <u>strictly monotone</u> if the inner product above is positive except when $x_1 = x_2$, and <u>maximal monotone</u> if the graph of T is not properly contained in the graph of any other monotone operator. The <u>inverse</u> of T is defined by $T^{-1}(y) := \{x | y \in T(x)\}$. For more information about such operators, see [2].

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and by the National Science Foundation under Grant No. MCS-7901066, Mod. 1.

2. AN OBSERVATION ABOUT SUMS OF MONOTONE OPERATORS.

Let F and G be monotone operators from R^n to itself. We shall be interested in the operator $(F + G)^{-1}$, which carries points $y \in R^n$ to points $x \in R^n$ such that $y \in F(x) + G(x)$ (if any such x exist). One reason for our interest in such operators is that problems in applications frequently occur in that form. One example, treated in more detail in Section 3, is the positive semidefinite linear complementarity problem: given an $n \times n$ positive semidefinite matrix A (i.e., a matrix A such that $\langle x,Ax\rangle \geqq 0$ for each $x \in R^n$) and a point $a \in R^n$, this problem asks for $x \in R^n$ such that

$$x \geqq 0,\ Ax + a \geqq 0,\ \langle x, Ax + a\rangle = 0\ , \tag{2.1}$$

where the inequalities in (2.1) are componentwise. If we define $F(x) := Ax + a$ and $G(x)$ to be the normal cone to the non-negative orthant R^n_+:

$$G(x) := \partial\psi_{R^n_+}(x) = \begin{cases} \{y \mid \text{ for each } z > 0,\ \langle z - x,y\rangle \leqq 0\} & \text{if } x \geqq 0 \\ \phi & \text{if } x \ngeqq 0\ , \end{cases}$$

$$= \begin{cases} \{y \leqq 0 \mid \langle y,x\rangle = 0\} & \text{if } x \geqq 0 \\ \phi & \text{if } x \ngeqq 0\ , \end{cases}$$

then (2.1) is equivalent to $0 \in F(x) + G(x)$, and therefore the set of x satisfying (2.1) is $(F + G)^{-1}(0)$. It is easy to verify that both F and G are monotone operators; in fact each is maximal, as one may show by using the tests described in [2, Ch. 2].

In order to establish various properties of $(F + G)^{-1}$, we shall impose upon F a special requirement that will make possible everything else we do in this paper. This requirement is that F^{-1} be strictly monotone. As one can see from the definitions in the introduction, this implies that if (x_1,y_1) and (x_2,y_2) belong to the graph of F, then either $\langle x_1 - x_2, y_1 - y_2\rangle > 0$ or

$y_1 = y_2$. Similar but somewhat stronger requirements have been used before by Gol'shtein and Tret'yakov [3] and by Poljak [7] for different purposes.

It is clear that if F^{-1} is strictly monotone, then F cannot be multivalued. However, the converse does not hold, as one can see by considering the skew linear operator on R^2 defined by $F(\xi_1,\xi_2) := (\xi_2,-\xi_1)$. This F is maximal monotone (its matrix, being skew, is trivially positive semidefinite), but $F^{-1}(=-F)$ is not strictly monotone (take $x_1 = 0$ and $x_2 \neq 0$). However, it is possible to show that if F is the gradient of any Lipschitz continuously differentiable convex function on R^n, then in fact F^{-1} is <u>strongly</u> monotone; that is, for some $\mu > 0$ and each $x_1,x_2 \in R^n$, $\langle x_1 - x_2, F(x_1) - F(x_2)\rangle \geq \mu\|F(x_1) - F(x_2)\|^2$. (See, e.g., [7]). Thus, there is already a wide class of possible candidates for F that will satisfy our basic assumption.

If we are looking for a point $x \in (F + G)^{-1}(y)$, we have to find some w with the property that $(x,w) \in F$ and $(x,y - w) \in G$. Of course, in general we cannot say anything about w, but if our special requirement on F is satisfied, then it turns out that there is just one w that satisfies these requirements. Further, once this w is identified we have a representation of $(F + G)^{-1}(y)$ as an intersection of inverse images of points under F and G separately. These facts are stated formally in the following proposition. We use $\operatorname{im}(F + G)$ to denote the <u>image</u> of $F + G$: that is, the set $\cup\{(F + G)(x) \mid x \in R^n\}$.

PROPOSITION 1: <u>Let</u> F <u>and</u> G <u>be monotone operators from</u> R^n <u>to itself, with</u> F^{-1} <u>strictly monotone. Then the operator</u> $\Phi := (I + G\circ F^{-1})^{-1}$ <u>is a single-valued function from</u> $\operatorname{im}(F + G)$ <u>to</u> R^n, <u>and for each</u> $y \in \operatorname{im}(F + G)$, <u>the point</u> $w = \Phi(y)$ <u>is the unique</u> w <u>such that for some</u> $x \in R^n$, $w \in F(x)$ <u>and</u> $y - w \in G(x)$. <u>Further, one has</u>

$$(F + G)^{-1} = [F^{-1}\circ\Phi] \cap [G^{-1}\circ(I - \Phi)] \ . \tag{2.2}$$

PROOF: We will show first that there is a unique w having the property stated, then that this w is given by Φ as defined in the proposition, and finally that the representation of $(F + G)^{-1}$ as an intersection is valid.

Choose any $y \in \text{im}\,(F + G)$; then there is some $x \in R^n$ with $y \in (F + G)(x) = F(x) + G(x)$. Thus, there is some $w \in F(x)$ with $y - w \in G(x)$. Now suppose that for some x' and w' we have $w' \in F(x')$ and $y - w' \in G(x')$. Then by monotonicity of F and G,

$$0 \leqq \langle x - x', w - w' \rangle \leqq \langle x - x', w - w' \rangle + \langle x - x', (y - w) - (y - w') \rangle = 0 ,$$

so that $\langle x - x', w - w' \rangle = 0$ and hence $w = w'$ by strict monotonicity of F^{-1}. Therefore w is unique. Now note that w is characterized by the fact that $w \in F(x)$ and $y - w \in G(x)$ for some x. This is equivalent to saying that $y - w \in (G \circ F^{-1})(w)$, which in turn is equivalent to $y \in (I + G \circ F^{-1})(w)$ and thus to $w \in (I + G \circ F^{-1})^{-1}(y) = \Phi(y)$. Our uniqueness proof shows that Φ is single-valued.

To establish (2.2), note that for any point y, if for some x and w we have $x \in F^{-1}(w) \cap G^{-1}(y - w)$, then $x \in (F + G)^{-1}(y)$. Thus $(F + G)^{-1} \supset [F^{-1} \circ \Phi] \cap [G^{-1} \circ (I - \Phi)]$. To establish the opposite inclusion, let $x \in (F + G)^{-1}(y)$. Our previous argument shows that $\Phi(y) \in F(x)$ and $y - \Phi(y) \in G(x)$. Thus $x \in F^{-1}[\Phi(y)] \cap G^{-1}[y - \Phi(y)] = [F^{-1} \circ \Phi] \cap [G^{-1} \circ (I - \Phi)](y)$, so (2.2) holds. This completes the proof.

Proposition 1 shows that Φ is a single-valued function from $\text{im}\,(F + G)$ to R^n. In special cases, this function is very familiar: for example, if we take $F = \lambda I$ for some $\lambda > 0$, then a routine computation shows that $\Phi = \lambda (G^{-1})_\lambda$, where T_λ denotes the Yosida approximation to an operator T [2]. Also, in this case the operator $I - \Phi$ is just the resolvent of G^{-1}. These operators are known to be (Lipschitz) continuous if G is maximal (and in that case $\text{im}\,(F + G) = R^n$; see [2]). Thus, we might ask whether in the general case Φ can be shown to be continuous, provided that we assume some reasonable conditions on F and G, such as maximality. The answer is yes, as we shall show next.

Before stating the continuity result, we observe that if F is maximal monotone and F^{-1} is strictly monotone, then the effective domain $\text{dom}\, F = \{x | F(x) \neq \phi\}$ is an open set. This is true since if x_0 is any boundary point of $\text{dom}\, F$, then either $F(x_0)$ is empty or it contains a half-line. The latter is

impossible since we have already seen that F must be single-valued (if nonempty), so it must be the case that $\operatorname{dom} F$ contains none of its boundary points, i.e., that it is open.

THEOREM 2: <u>Suppose that</u> F <u>and</u> G <u>are maximal monotone operators from</u> R^n <u>to itself, and that</u> F^{-1} <u>is strictly monotone. Then</u> Φ <u>is continuous on</u> $\operatorname{int} \operatorname{im} (F + G)$.

PROOF: The conclusion is vacuously true if $\operatorname{int} \operatorname{im} (F + G) = \phi$, so we may assume that $\operatorname{im} (F + G)$ has a nonempty interior. This shows in particular that $\operatorname{dom} F \cap \operatorname{dom} G \neq \phi$. It can be shown, using the maximal monotonicity of F and G, that $\operatorname{ri} \operatorname{dom} F \subset \operatorname{dom} F \subset \operatorname{cl} \operatorname{dom} F$ and $\operatorname{ri} \operatorname{dom} G \subset \operatorname{dom} G \subset \operatorname{cl} \operatorname{dom} G$, where the symbol ri denotes the interior of a set relative to its affine hull; further the outer members of these inclusions are convex. However, this implies that $\operatorname{dom} F$ itself is an open convex set, and it is an easy exercise in convex analysis to show from this that we actually have $(\operatorname{int}) \operatorname{dom} F \cap \operatorname{ri} \operatorname{dom} G \neq \phi$. By a standard result, we now find that $F + G$ is maximal monotone and that $(F + G)^{-1}$ is locally bounded at each point of $\operatorname{int} \operatorname{im} (F + G)$.

The rest of the proof will consist in showing that Φ is closed at a point of $\operatorname{int} \operatorname{im} (F + G)$, then that it is locally bounded there. These two facts, together with the fact that Φ is single-valued, immediately imply continuity.

To show that Φ is closed at a point $y_0 \in \operatorname{int} \operatorname{im} (F + G)$, choose a neighborhood N of y_0 small enough so that $N \subset \operatorname{im} (F + G)$ and $(F + G)^{-1}(N)$ is bounded. If $\{y_k\}$ is a sequence in N converging to y_0, and if $\{\Phi(y_k)\}$ converges to some z_0, we want to show that $z_0 = \Phi(y_0)$. Since $N \subset \operatorname{im} (F + G)$, for each k there is some x_k with $y_k \in F(x_k) + G(x_k)$; further, $\{x_k\}$ is a bounded sequence, so with no loss of generality we can assume that $\{x_k\}$ converges to some x_0. For each k, we have $\Phi(y_k) \in F(x_k)$ and $y_k - \Phi(y_k) \in G(x_k)$; since F and G, being maximal monotone, are closed operators we find that $z_0 \in F(x_0)$ and $y_0 - z_0 \in G(x_0)$. Applying Proposition 1, we have $z_0 = \Phi(y_0)$ as desired.

For local boundedness, let $y_0 \in \operatorname{int} \operatorname{im} (F + G)$ and suppose there is a sequence $\{y_k\} \subset \operatorname{im} (F + G)$ converging to y_0 with $\|\Phi(y_k)\| \to +\infty$. Without loss of

generality we can suppose that $\Phi(y_k)/\|\Phi(y_k)\|$ converges to some h. However, we noted previously that $(F + G)^{-1}$ is locally bounded at y_0, so we can suppose that there is a sequence $\{x_k\}$, converging to some x_0, with $x_k \in (F + G)^{-1}(y_k)$ for each k. But then for each pair (x,f) in the graph of F and each (z,g) in the graph of G, we have by monotonicity that the inner products $\langle x - x_k, f - \Phi(y_k) \rangle$ and $\langle z - x_k, g - y_k + \Phi(y_k)\rangle$ are non-negative. Dividing these expressions by $\|\Phi(y_k)\|$ and taking the limit, we have for each $x \in \operatorname{dom} F$ and each $z \in \operatorname{dom} G$,

$$\langle x - x_0, h \rangle \leq 0 \leq \langle z - x_0, h \rangle .$$

The set $\operatorname{dom} F$, being open, cannot be contained in the hyperplane $H := \{w \mid \langle w,h\rangle = \langle x_0,h\rangle\}$. Therefore H properly separates $\operatorname{dom} F$ and $\operatorname{dom} G$, contradicting the fact that $\operatorname{int} \operatorname{dom} F \cap \operatorname{ri} \operatorname{dom} G \neq \phi$ [12, Th. 11.3]. Accordingly, our assumption about $\{y_k\}$ was wrong, so that Φ is locally bounded at y_0. This completes the proof.

In this section we have developed some general properties of Φ, but we have not shown why it is of any interest. In the next section, we show how the use of Φ provides insight into the structure of solution sets of linear generalized equations (such as the complementarity problem considered earlier).

3. APPLICATION TO LINEAR GENERALIZED EQUATIONS.

In this section we shall apply the mapping Φ to identify the structure of the solution set of a linear generalized equation of the form

$$o \in Ax + a + M(x) , \qquad (3.1)$$

where A is a positive semidefinite linear operator from R^n to itself, $a \in R^n$, and M is a monotone operator from R^n to itself. For more information about generalized equations and their applications, see [4, 5, 6, 8, 9, 10, 11]

Our first result is a structure theorem for the solution set of (3.1). We use the symbol "ker" to denote the kernel of a linear operator.

THEOREM 3: Suppose that in (3.1) A is positive semidefinite and M is monotone. Assume that $x_0 \in R^n$ is a solution of (3.1) and denote the symmetric part of A by S and the skew part by K. Then the solution set of (3.1) is

$$\{x_0 + \ker S\} \cap \{x \mid 0 \in Kx + (Sx_0 + a) + M(x)\} . \qquad (3.2)$$

If C is a closed convex cone in R^n and $M = \partial\psi_C$, then (3.2) becomes the set of $x \in R^n$ such that

$$\begin{aligned} &Ax + a \in C^* \\ &x \in C \\ &\langle x, Sx_0 + a \rangle = 0 \\ &S(x - x_0) = 0 , \end{aligned} \qquad (3.3)$$

where C^* is the dual cone of C : $C^* = \{z \in R^n \mid \langle x,c \rangle \geq 0$ for each $c \in C\}$.

PROOF: We define two monotone operators F and G from R^n to itself by

$$F(x) := Sx, \; G(x) := Kx + a + M(x) .$$

Evidently x satisfies (3.1) if and only if $x \in (F + G)^{-1}(0)$. Also, it is well known that $\langle z, Sz \rangle = 0$ only if $Sz = 0$, so the operator F^{-1} is strictly monotone. As x_0 satisfies (3.1), we have $Sx_0 = F(x_0)$ and $0 - Sx_0 \in Kx_0 + a + M(x_0) = G(x_0)$. Applying Proposition 1, we find that $\Phi(0) = Sx_0$ and

$$\begin{aligned} (F + G)^{-1}(0) &= \{x \mid Sx = Sx_0\} \cap \{x \mid 0 - Sx_0 \in Kx + a + M(x)\} \\ &= \{x_0 + \ker S\} \cap \{x \mid 0 \in Kx + (Sx_0 + a) + M(x)\} , \end{aligned}$$

which proves (3.2). Now suppose that $M = \partial\psi_C$ for some nonempty closed convex cone C. In that case, it is well known that

$$\partial\psi_C(x) = \begin{cases} \{-y \mid y \in C^*, \langle x,y \rangle = 0\} & \text{if } x \in C, \\ \phi & \text{if } x \notin C, \end{cases}$$

so that the requirement $0 \in Kx + (Sx_0 + a) + M(x)$ becomes

$$Kx + (Sx_0 + a) \in C^* , \tag{3.4}$$

$$x \in C , \tag{3.5}$$

$$\langle x, Kx + (Sx_0 + a) \rangle = 0 , \tag{3.6}$$

and we also have

$$S(x - x_0) = 0 . \tag{3.7}$$

However, (3.7) and (3.4) together are equivalent to (3.7) and

$$Ax + a \in C^* . \tag{3.8}$$

Also, $\langle x,Kx \rangle = 0$ because K is skew. Thus (3.6) is equivalent to

$$\langle x, Sx_0 + a \rangle = 0 . \tag{3.9}$$

The conditions (3.8), (3.5), (3.9) and (3.7) then yield (3.3), and this completes the proof.

We observe that if C is polyhedral, then (3.3) shows that its solutions form a polyhedral convex set.

In the particular case $C = R^n_+$, the conditions (3.3) reduce to those given by Adler and Gale [1] except that the single equation (3.9) replaces the two equations numbered (11) and (12) in their paper; in our notation these two equations are $\langle x_0, Ax + a \rangle = 0$ and $\langle x, Ax_0 + a \rangle = 0$. It can be shown directly that the two sets of conditions are equivalent, and of course the general characterizations given here and in [1] also show this equivalence.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER 2177	2. GOVT ACCESSION NO. AD-A099365	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) INVERSE SUMS OF MONOTONE OPERATORS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Stephen M. Robinson		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-7901066, Mod. 1
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 5 - Operations Research
11. CONTROLLING OFFICE NAME AND ADDRESS (See Item 18 below)		12. REPORT DATE February 1981
		13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Monotone operators, generalized equations, linear complementarity problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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